
Quantum mechanics II, Chapter 4 : Fermions and Bosons

TA : Slimane Thabet, Sofia Brizigotti, Alba Miren Taddei, Reyhaneh Aghaei Saem, Mehrad Sahebi, Ricard Puig, Sacha Lerch

Problem 1 : Warm up

Boson. Imagine we have two photons in two spatial modes labelled $|\mathbf{k}_1\rangle$ or $|\mathbf{k}_2\rangle$ with horizontal and/or vertical polarizations. Which of these is a valid symmetric state for the two photons ?

$$(1) |\psi_1\rangle = \frac{1}{\sqrt{2}} (|\mathbf{k}_1, \mathbf{k}_2\rangle + |\mathbf{k}_2, \mathbf{k}_1\rangle) \otimes \frac{1}{\sqrt{2}} (|H, V\rangle + |V, H\rangle)$$

To see if the state is symmetric, we can apply the permutation operator \mathbb{P}_{12} on it and see if the eigenvalue of that state is $+1$ or -1 .

$$\mathbb{P}_{12} |\psi_1\rangle = \frac{1}{2} (|\mathbf{k}_2, \mathbf{k}_1\rangle \otimes |V, H\rangle + |\mathbf{k}_2, \mathbf{k}_1\rangle \otimes |H, V\rangle + |\mathbf{k}_1, \mathbf{k}_2\rangle \otimes |V, H\rangle + |\mathbf{k}_1, \mathbf{k}_2\rangle \otimes |H, V\rangle) \quad (1)$$

$$= \frac{1}{\sqrt{2}} (|\mathbf{k}_1, \mathbf{k}_2\rangle + |\mathbf{k}_2, \mathbf{k}_1\rangle) \otimes \frac{1}{\sqrt{2}} (|H, V\rangle + |V, H\rangle) = |\psi_1\rangle \quad (2)$$

So $|\psi_1\rangle$ is symmetric.

$$(2) |\psi_2\rangle = \frac{1}{\sqrt{2}} (|\mathbf{k}_1, \mathbf{k}_2\rangle - |\mathbf{k}_2, \mathbf{k}_1\rangle) \otimes \frac{1}{\sqrt{2}} (|H, V\rangle - |V, H\rangle)$$

We can do the same thing we did for the first part for the following states and whether they are symmetric. $|\psi_2\rangle$ is symmetric.

$$(3) |\psi_3\rangle = |\mathbf{k}_1, \mathbf{k}_2\rangle \otimes \frac{1}{\sqrt{2}} (|H, V\rangle + |V, H\rangle)$$

None of them.

$$(4) |\psi_4\rangle = \frac{1}{\sqrt{2}} (|\mathbf{k}_1, \mathbf{k}_2\rangle + |\mathbf{k}_2, \mathbf{k}_1\rangle) \otimes |H, V\rangle$$

None of them.

Fermions. Consider two electrons in Helium which are in two different orbital states (one in the ground state, one in an excited state), such as we labelled $|a\rangle$, $|b\rangle$ above with up and/or down spins. Which 2 of the following 6 states are possible states of the two electrons ? Justify.

$$(1) |\psi_1\rangle = \frac{1}{\sqrt{2}} (|a, b\rangle + |b, a\rangle) \otimes \frac{1}{\sqrt{2}} (|\uparrow, \uparrow\rangle + |\downarrow, \downarrow\rangle)$$

For the Fermions, the total state has to be asymmetric under the exchange of the two particles.

$$\mathbb{P}_{12} |\psi_1\rangle = |\psi_1\rangle \quad (3)$$

So the state is symmetric and not possible for two Fermions.

$$(2) |\psi_2\rangle = \frac{1}{\sqrt{2}} (|a, b\rangle + |b, a\rangle) \otimes \frac{1}{\sqrt{2}} (|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle)$$

Symmetric.

$$(3) |\psi_3\rangle = \frac{1}{\sqrt{2}} (|a, b\rangle - |b, a\rangle) \otimes \frac{1}{\sqrt{2}} (|\uparrow, \uparrow\rangle + |\downarrow, \downarrow\rangle)$$

Antisymmetric.

$$(4) |\psi_4\rangle = \frac{1}{\sqrt{2}} (|a, b\rangle - |b, a\rangle) \otimes \frac{1}{\sqrt{2}} (|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle)$$

Symmetric.

$$(5) |\psi_5\rangle = \frac{1}{\sqrt{2}}(|a, b\rangle - |b, a\rangle) \otimes \frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle + |\downarrow, \uparrow\rangle)$$

Antisymmetric.

$$(6) |\psi_6\rangle = |a, b\rangle \otimes \frac{1}{\sqrt{2}}(|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle)$$

None of them.

Problem 2 : Bosonic wavefunction

Consider the possible basis states for a system of n Bosons. It can be written as

$$|\psi_{\mathbf{x}}\rangle = \mathcal{N} \sum_{\mathbb{P} \in \mathcal{S}_n} \mathbb{P}|x_1, x_2, \dots, x_n\rangle = \mathcal{N} \sum_{\mathbb{P} \in \mathcal{S}_n} |x_{\mathbb{P}(1)}\rangle |x_{\mathbb{P}(2)}\rangle \dots |x_{\mathbb{P}(n)}\rangle, \quad (4)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$, \mathcal{N} is a normalization factor, and \mathcal{S}_n is the symmetric group on n elements.

Prove that the normalization factor is

$$\mathcal{N} = \frac{1}{\sqrt{n!} \sqrt{\prod_k n_k!}} \quad (5)$$

where n_k is repeated entries and $\sum_k n_k = n$.

To find the normalization constant, we compute the norm of the state and set it equal to 1 :

$$1 = \langle \psi_{\mathbf{x}} | \psi_{\mathbf{x}} \rangle \quad (6)$$

$$= \mathcal{N}^2 \sum_{\mathbb{P} \in \mathcal{S}_n} \sum_{\mathbb{P}' \in \mathcal{S}_n} \langle x_{\mathbb{P}(1)} | x_{\mathbb{P}'(1)} \rangle \langle x_{\mathbb{P}(2)} | x_{\mathbb{P}'(2)} \rangle \langle x_{\mathbb{P}(3)} | x_{\mathbb{P}'(3)} \rangle \dots \langle x_{\mathbb{P}(n)} | x_{\mathbb{P}'(n)} \rangle \quad (7)$$

$$= \mathcal{N}^2 \sum_{\mathbb{P} \in \mathcal{S}_n} \sum_{\mathbb{P}' \in \mathcal{S}_n} \langle x_1 | x_{\mathbb{P}' \circ \mathbb{P}^{-1}(1)} \rangle \langle x_2 | x_{\mathbb{P}' \circ \mathbb{P}^{-1}(2)} \rangle \langle x_3 | x_{\mathbb{P}' \circ \mathbb{P}^{-1}(3)} \rangle \dots \langle x_n | x_{\mathbb{P}' \circ \mathbb{P}^{-1}(n)} \rangle \quad (8)$$

$$= \mathcal{N}^2 n! \sum_{\mathbb{P} \in \mathcal{S}_n} \langle x_1 | x_{\mathbb{P}(1)} \rangle \langle x_2 | x_{\mathbb{P}(2)} \rangle \langle x_3 | x_{\mathbb{P}(3)} \rangle \dots \langle x_n | x_{\mathbb{P}(n)} \rangle, \quad (9)$$

where in the third line we rearranged the terms and in the last term used that $\sum_{\mathbb{P}, \mathbb{P}'} \mathbb{P} \circ \mathbb{P}' = n! \sum_{\mathbb{P}} \mathbb{P}$. In the case, where all the single-particle states are orthogonal (i.e. every state has occupation number $n_k = 1$), only the term $\mathbb{P} = \mathbb{1}$ is non-zero and the claim is verified.

If some states are multiply occupied, $|x_i\rangle = |x_j\rangle$ for some $i \neq j$ and other terms in the sum have a non-zero contribution. In fact, we find that all the terms evaluate to 1, where \mathbb{P} permutes the indices in a way that $\forall i$ we have $|x_i\rangle = |x_{\mathbb{P}(i)}\rangle$. This is the case, if

$$\mathbb{P} \in S(n_1) \times S(n_2) \times \dots \times S(n_K) \subset S(n), \quad (10)$$

i.e. \mathbb{P} only permutes particles occupying the same state. Here, n_k denotes the occupation number of the K orthogonal single-particle states. The size of each permutation group $S(n_k)$ is $n_k!$ and thus we find

$$1 = \mathcal{N}^2 n! |S(n_1) \times S(n_2) \times \dots \times S(n_K)| = \mathcal{N}^2 n! \prod_{k=1}^K n_k!. \quad (11)$$

Then we have

$$\mathcal{N} = \frac{1}{\sqrt{n!} \sqrt{\prod_k n_k!}}. \quad (12)$$

Problem 3 : Eigenspectra of fermions and bosons

We are familiar with simple harmonic oscillators from Quantum Physics 1. A particle with mass m in one-dimensional harmonic potential $V(x) = \frac{1}{2}m\omega^2x^2$ has the following eigenenergies.

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right), \quad n = 0, 1, \dots \quad (13)$$

and we call $\varphi_n(x)$ the associated eigenfunctions which we suppose normalized.

Now consider that we want to study a system of identical particles that do not interact so that we can consider the total Hamiltonian of these systems as a sum of one particle Hamiltonian.

Bosons

We take the particles as Bosons of zero spin here. So the eigenfunction of Bosons needs the spatial part which must be symmetrical and we know the one particle case of them from $\varphi_n(x)$.

1. Consider the case of 2 bosons. Find the first 3 energy levels, their degeneracy, and the associated eigenfunctions.

(Hint : Use $\varphi_n(x)$ s to make two-particle symmetric wavefunctions.)

In this section, we assume that the particles are spinless bosons. The wave function of the system therefore consists solely of a spatial part, which must be symmetrical. For a system of two bosons, the total energy of the system is given by

$$E = E_{n_1} + E_{n_2}, \quad (14)$$

where each E_{n_i} is given by $E_{n_i} = \hbar\omega \left(n_i + \frac{1}{2} \right)$.

The lowest energy level is the one with both bosons in the $n_1 = n_2 = 0$ level. We can have this energy just with this combination of n_i s and so the degeneracy is 1. The ground state energy $E_{\text{ground state}}$ is

$$E_{\text{ground state}} = E_0 + E_0 = \hbar\omega \quad (15)$$

and the symmetric eigenfunctions is

$$\psi_{\text{ground state}}(x_1, x_2) = \varphi_0(x_1)\varphi_0(x_2) \quad (16)$$

The first excited state corresponds to the state with a boson in $n = 0$, and the other one in $n = 1$. The eigenenergy is

$$E_{\text{1st excited state}} = E_0 + E_1 = 2\hbar\omega \quad (17)$$

The symmetric eigenfunction is

$$\psi_{\text{1st excited state}}(x_1, x_2) = \frac{1}{\sqrt{2}} [\varphi_0(x_1)\varphi_1(x_2) + \varphi_1(x_1)\varphi_0(x_2)] \quad (18)$$

and the degeneracy is still 1.

The second excited state corresponds to either the boson in $n = 0$, and the other in $n = 2$, or to the state with both bosons in $n = 1$. The eigenenergy is

$$E_{\text{2nd excited state}} = E_0 + E_2 = E_1 + E_1 = 3\hbar\omega \quad (19)$$

and the symmetric eigenfunction in the case of bosons in two different energy levels

$$\psi_{\text{2nd excited state}}^{(1)}(x_1, x_2) = \frac{1}{\sqrt{2}} [\varphi_0(x_1)\varphi_2(x_2) + \varphi_2(x_1)\varphi_0(x_2)] \quad (20)$$

and in the case where bosons are in the same energy levels

$$\psi_{2\text{nd excited state}}^{(2)}(x_1, x_2) = \varphi_1(x_1)\varphi_1(x_2). \quad (21)$$

It means that we have two eigenfunctions for one energy so the degeneracy is 2.

2. We are now interested in the case of 3 bosons. Find the first 3 energy levels, their degeneracy and the associated eigenfunctions.

We now look at the 3 boson case. For a system of three bosons, the total energy of the system is given by

$$E = E_{n_1} + E_{n_2} + E_{n_3}. \quad (22)$$

The lowest energy level is the one with the 3 bosons in the energy level E_0 . The eigenenergy is

$$E_{\text{ground state}} = E_0 + E_0 + E_0 = \frac{3}{2}\hbar\omega \quad (23)$$

and the symmetric eigenfunction is

$$\psi_{\text{ground state}}(x_1, x_2, x_3) = \varphi_0(x_1)\varphi_0(x_2)\varphi_0(x_3), \quad (24)$$

so the degeneracy is 1,

The first excited state corresponds to the state with 2 bosons in E_0 , and the third one in E_1 . The eigenenergy is

$$E_{1\text{st excited state}} = E_0 + E_0 + E_1 = \frac{5}{2}\hbar\omega \quad (25)$$

The symmetric eigenfunction is

$$\psi_{1\text{st excited state}}(x_1, x_2, x_3) = \frac{1}{\sqrt{3}} [\varphi_0(x_1)\varphi_0(x_2)\varphi_1(x_3) + \varphi_0(x_1)\varphi_1(x_2)\varphi_0(x_3) + \varphi_1(x_1)\varphi_0(x_2)\varphi_0(x_3)] \quad (26)$$

and the degeneracy is still 1.

The second excited state corresponds to the state with one Boson in E_0 , and two Bosons in E_1 or two Bosons in E_0 , and one Boson in E_2 . The eigenenergy is

$$E_{2\text{nd excited state}} = E_0 + E_0 + E_2 = E_0 + E_1 + E_1 = \frac{7}{2}\hbar\omega \quad (27)$$

The symmetric eigenfunction in the case of two Bosons in E_0 , and one Boson in E_2 is

$$\psi_{2\text{nd excited state}}^{(1)}(x_1, x_2, x_3) = \frac{1}{\sqrt{3}} [\varphi_0(x_1)\varphi_0(x_2)\varphi_2(x_3) + \varphi_0(x_1)\varphi_2(x_2)\varphi_0(x_3) + \varphi_2(x_1)\varphi_0(x_2)\varphi_0(x_3)] \quad (28)$$

The symmetric eigenfunction in the case of one Boson in E_0 , and two Bosons in E_1 is

$$\psi_{2\text{nd excited state}}^{(2)}(x_1, x_2, x_3) = \frac{1}{\sqrt{3}} [\varphi_0(x_1)\varphi_1(x_2)\varphi_1(x_3) + \varphi_1(x_1)\varphi_0(x_2)\varphi_1(x_3) + \varphi_1(x_1)\varphi_1(x_2)\varphi_0(x_3)] \quad (29)$$

and the degeneracy is 2.

Fermions

In this section, we are interested in the case of two electrons, i.e. two fermions of spin of spin 1/2, and we are looking for the eigenfunctions of two particles in the form

$$\psi(x_1, x_2, s_1, s_2) = \varphi(x_1, x_2)\chi(s_1, s_2), \quad (30)$$

where x_i denote the position of the particles and $s_i \in \{+\frac{1}{2}, -\frac{1}{2}\}$ their spins. $\varphi(x_1, x_2)$ is the spatial part and $\chi(s_1, s_2)$ is the spin part of the wavefunction.

1. $\chi(s_1, s_2)$ means that the first particle has spin s_1 and the second particle has spin s_2 . What is the dimension of the space of functions $\chi(s_1, s_2)$?

The wave functions live in the Hilbert space spanned by the tensor product of two spin 1/2 particles $\mathcal{H} = \mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}}$. Thus $\dim \mathcal{H} = (\dim \mathcal{H}_{\frac{1}{2}})^2 = 2^2 = 4$

2. We define in this space, the operators \mathbb{P} (permutation) and S^z (total spin in the \hat{z} direction) by :

$$\begin{aligned}\mathbb{P}\chi(s_1, s_2) &= \chi(s_2, s_1) \\ S^z\chi(s_1, s_2) &= (s_1 + s_2)\chi(s_1, s_2)\end{aligned}\tag{31}$$

In other words, we know that $\chi(s_1, s_2)$ is the eigenfunction of S^z with eigenvalue $s_1 + s_2$.

- (a) Demonstrate that we can diagonalize \mathbb{P} and S^z in a common basis.

We show that the two operators commute :

$$\mathbb{P}S^z\chi(s_1, s_2) = \mathbb{P}(s_1 + s_2)\chi(s_1, s_2) = (s_1 + s_2)\chi(s_2, s_1) = S^z\chi(s_2, s_1) = S^z\mathbb{P}\chi(s_1, s_2),$$

where we used that $s_1, s_2 \in \mathbb{R}$ commute.

- (b) Using the previous result find the common eigenstates of \mathbb{P} and S^z .

(Hint : S^z is already diagonal in the basis of $\chi(s_1, s_2)$ so try to diagonalize \mathbb{P} in this basis.)

To find the mutual eigenfunctions of \mathbb{P} and S^z , we diagonalise \mathbb{P} in each eigenspace of S^z . The eigenspace of S^z is given by $\chi(s_1, s_2)$, here we write all eigenvectors and their eigenvalues.

The first eigenvalue is 1 with the following eigenvector.

$$S^z\chi\left(+\frac{1}{2}, +\frac{1}{2}\right) = \chi\left(+\frac{1}{2}, +\frac{1}{2}\right)\tag{32}$$

The second eigenvalue is -1 with the following eigenvector.

$$S^z\chi\left(-\frac{1}{2}, -\frac{1}{2}\right) = -\chi\left(-\frac{1}{2}, -\frac{1}{2}\right)\tag{33}$$

The third eigenvalue is 0 which has degeneracy 2 and the following vectors are its eigenvectors.

$$S^z\chi\left(+\frac{1}{2}, -\frac{1}{2}\right) = 0\tag{34}$$

$$S^z\chi\left(-\frac{1}{2}, +\frac{1}{2}\right) = 0\tag{35}$$

Let's also find the effect of \mathbb{P} on this basis.

$$\mathbb{P}\chi\left(+\frac{1}{2}, +\frac{1}{2}\right) = \chi\left(+\frac{1}{2}, +\frac{1}{2}\right)\tag{36}$$

$$\mathbb{P}\chi\left(-\frac{1}{2}, -\frac{1}{2}\right) = \chi\left(-\frac{1}{2}, -\frac{1}{2}\right)\tag{37}$$

$$\mathbb{P}\chi\left(+\frac{1}{2}, -\frac{1}{2}\right) = \chi\left(-\frac{1}{2}, +\frac{1}{2}\right)\tag{38}$$

$$\mathbb{P}\chi\left(-\frac{1}{2}, +\frac{1}{2}\right) = \chi\left(+\frac{1}{2}, -\frac{1}{2}\right)\tag{39}$$

Here we can see that $\chi(+\frac{1}{2}, +\frac{1}{2})$ and $\chi(-\frac{1}{2}, -\frac{1}{2})$ are also eigenvectors for \mathbb{P} . So we just need to diagonalize \mathbb{P} in the subspace of $\{\chi(+\frac{1}{2}, -\frac{1}{2}), \chi(-\frac{1}{2}, +\frac{1}{2})\}$. We can see that \mathbb{P} maps $\chi(+\frac{1}{2}, -\frac{1}{2})$ to $\chi(-\frac{1}{2}, +\frac{1}{2})$ and vice-versa. Then two eigenvectors of \mathbb{P} in this subspace are namely the symmetric eigenstate $\frac{1}{\sqrt{2}}(\chi(+\frac{1}{2}, -\frac{1}{2}) + \chi(-\frac{1}{2}, +\frac{1}{2}))$ and the antisymmetric eigenstate $\frac{1}{\sqrt{2}}(\chi(+\frac{1}{2}, -\frac{1}{2}) - \chi(-\frac{1}{2}, +\frac{1}{2}))$. Finally, the common eigenstates of \mathbb{P} and S^z are as follows.

$$\chi_{++}(s_1, s_2) = \chi(+\frac{1}{2}, +\frac{1}{2}) \quad (40)$$

$$\chi_{--}(s_1, s_2) = \chi(-\frac{1}{2}, -\frac{1}{2}) \quad (41)$$

$$\chi_{sym}(s_1, s_2) = \frac{1}{\sqrt{2}} \left(\chi(+\frac{1}{2}, -\frac{1}{2}) + \chi(-\frac{1}{2}, +\frac{1}{2}) \right) \quad (42)$$

$$\chi_{antisym}(s_1, s_2) = \frac{1}{\sqrt{2}} \left(\chi(+\frac{1}{2}, -\frac{1}{2}) - \chi(-\frac{1}{2}, +\frac{1}{2}) \right) \quad (43)$$

3. We now come back to the two electrons in a harmonic potential problem. Find the first 3 energy levels, their degeneracy, and the associated eigenfunctions.

(Hint : Find antisymmetric wavefunctions by trying the possible combination of spatial and spin parts.)

We know that for two Fermions, the total wave function needs to be antisymmetric. This means that the total wave function is composed either of a symmetric spatial part and antisymmetric spin part or an antisymmetric spatial part and symmetric spin part.

Similar to the Bosons part of this question, for a system of 2 electrons, the total energy of the system is given by

$$E = E_{n_1} + E_{n_2}. \quad (44)$$

The lowest energy level is achieved when both electrons are in $n = 0$.

$$E_{\text{ground state}} = E_0 + E_0 = \hbar\omega \quad (45)$$

In this case, the spatial part $\varphi_{\text{ground state}}(x_1, x_2) = \varphi_0(x_1)\varphi_0(x_2)$ is symmetric so the spin part needs to be antisymmetric. With the above derivations, we can thus write

$$\psi_{\text{ground state}}(x_1, x_2, s_1, s_2) = \varphi_0(x_1)\varphi_0(x_2) \chi_{antisym}(s_1, s_2) \quad (46)$$

which has degeneracy 1.

The first excited state The first excited energy $E_{1\text{st excited state}}$ is achieved when one electron occupies the first excited state $n = 1$ and the other one is in $n = 0$.

$$E_{1\text{st excited state}} = E_0 + E_1 = 2\hbar\omega \quad (47)$$

To achieve antisymmetry, we can choose either the spin or spatial part to be antisymmetric.

The solutions are

$$\psi_{1\text{st excited state}}^{(1)}(x_1, x_2, s_1, s_2) = \frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_1(x_2) + \varphi_0(x_2)\varphi_1(x_1)) \chi_{antisym}(s_1, s_2) \quad (48)$$

$$\psi_{1\text{st excited state}}^{(2)}(x_1, x_2, s_1, s_2) = \frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_1(x_2) - \varphi_0(x_2)\varphi_1(x_1)) \chi_{sym}(s_1, s_2) \quad (49)$$

$$\psi_{1\text{st excited state}}^{(3)}(x_1, x_2, s_1, s_2) = \frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_1(x_2) - \varphi_0(x_2)\varphi_1(x_1)) \chi_{++}(s_1, s_2) \quad (50)$$

$$\psi_{1\text{st excited state}}^{(4)}(x_1, x_2, s_1, s_2) = \frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_1(x_2) - \varphi_0(x_2)\varphi_1(x_1)) \chi_{--}(s_1, s_2) \quad (51)$$

$$(52)$$

so the degeneracy of the first excited state is 4.

The second excited state The second excited energy $E_{2\text{nd excited state}}$ is achieved when one electron occupies the second excited state $n = 2$ and the other one is in $n = 0$ or when both of the electrons are in the first excited state $n = 1$.

$$E_{2\text{nd excited state}} = E_0 + E_2 = E_1 + E_1 = 3\hbar\omega \quad (53)$$

To achieve antisymmetry, we can choose either the spin or spatial part to be antisymmetric. Let's first consider the case that both electrons are in $n = 1$. The solution is

$$\psi_{2\text{nd excited state}}^{(1)}(x_1, x_2, s_1, s_2) = \varphi_1(x_1)\varphi_1(x_2) \chi_{antisym}(s_1, s_2) \quad (54)$$

The second case is when one electron occupies the second excited state $n = 2$ and the other one is in $n = 0$. the solutions are

$$\psi_{2\text{nd excited state}}^{(2)}(x_1, x_2, s_1, s_2) = \frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_2(x_2) + \varphi_0(x_2)\varphi_2(x_1)) \chi_{antisym}(s_1, s_2)$$

$$\psi_{2\text{nd excited state}}^{(3)}(x_1, x_2, s_1, s_2) = \frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_2(x_2) - \varphi_0(x_2)\varphi_2(x_1)) \chi_{sym}(s_1, s_2)$$

$$\psi_{2\text{nd excited state}}^{(4)}(x_1, x_2, s_1, s_2) = \frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_2(x_2) - \varphi_0(x_2)\varphi_2(x_1)) \chi_{++}(s_1, s_2)$$

$$\psi_{2\text{nd excited state}}^{(5)}(x_1, x_2, s_1, s_2) = \frac{1}{\sqrt{2}}(\varphi_0(x_1)\varphi_2(x_2) - \varphi_0(x_2)\varphi_2(x_1)) \chi_{--}(s_1, s_2)$$

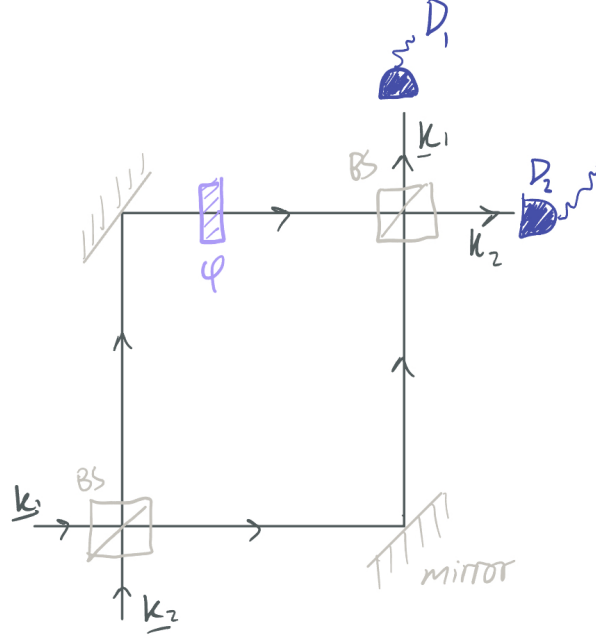
so the degeneracy of the second excited state is 5.

Problem 4 : Mach-Zehnder interferometer

The Mach-Zehnder interferometer consists of two 50 :50 beamsplitters with a phase shifter in one arm, arranged as follows :

When working in the second quantization it is often helpful to work in the Heisenberg picture and consider the action of any unitary process on the creation and annihilation operators rather than on a given state directly. The phase shift of φ in the upper arm in the Heisenberg picture acts as $\hat{a}_{k_2}^\dagger \rightarrow e^{i\varphi}\hat{a}_{k_2}^\dagger$. It follows that the total unitary matrix on the mode operators describing the apparatus is

$$U = U_{BS2} U_{phaseshift} U_{BS1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (55)$$



Where $U_{BS2} = U_{BS1}^\dagger$ and U_{BS1} is the effect of beamsplitter on the mode operators $(\hat{a}_{k_1}^\dagger, \hat{a}_{k_2}^\dagger)$. In other words, we know that in the Heisenberg picture, the creation operators are changed as follows

$$\hat{a}_{k_1}^\dagger \xrightarrow{BS1} \frac{1}{\sqrt{2}}(\hat{a}_{k_1}^\dagger + \hat{a}_{k_2}^\dagger), \quad (56)$$

$$\hat{a}_{k_2}^\dagger \xrightarrow{BS1} \frac{1}{\sqrt{2}}(\hat{a}_{k_1}^\dagger - \hat{a}_{k_2}^\dagger). \quad (57)$$

So we can write it in the following way

$$\begin{pmatrix} \hat{a}_{k_1}^\dagger \\ \hat{a}_{k_2}^\dagger \end{pmatrix} \xrightarrow{BS1} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \hat{a}_{k_1}^\dagger \\ \hat{a}_{k_2}^\dagger \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\hat{a}_{k_1}^\dagger + \hat{a}_{k_2}^\dagger) \\ \frac{1}{\sqrt{2}}(\hat{a}_{k_1}^\dagger - \hat{a}_{k_2}^\dagger) \end{pmatrix}. \quad (58)$$

We can also see the same calculations to find the unitary matrix of the phase shift.

$$\hat{a}_{k_1}^\dagger \xrightarrow{phase} \hat{a}_{k_1}^\dagger \quad (59)$$

$$\hat{a}_{k_2}^\dagger \xrightarrow{phase} e^{i\varphi} \hat{a}_{k_2}^\dagger. \quad (60)$$

So we can write it in the following way

$$\begin{pmatrix} \hat{a}_{k_1}^\dagger \\ \hat{a}_{k_2}^\dagger \end{pmatrix} \xrightarrow{phase} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \begin{pmatrix} \hat{a}_{k_1}^\dagger \\ \hat{a}_{k_2}^\dagger \end{pmatrix} = \begin{pmatrix} \hat{a}_{k_1}^\dagger \\ e^{i\varphi} \hat{a}_{k_2}^\dagger \end{pmatrix}. \quad (61)$$

- (i) Suppose one photon is input into the mode \mathbf{k}_1 . What is the probability as a function of ϕ of finding it at each of the detectors D_1 and D_2 ?

Consider that $|0\rangle_{\mathbf{k}_1}$ and $|0\rangle_{\mathbf{k}_2}$ are the vacuum states of modes \mathbf{k}_1 and \mathbf{k}_2 . Then $|n\rangle_{\mathbf{k}_i}$ means that we have n photons in mode \mathbf{k}_i .

We suppose that we have one photon in the mode \mathbf{k}_1 as input. So it means that our initial state is $\hat{a}_{k_1}^\dagger |0\rangle_{\mathbf{k}_1} |0\rangle_{\mathbf{k}_2}$. To find the final state after the effect of beamsplitters and the phase

shifter, we work in the Heisenberg picture and find the evolution of the creation operator. To do this, we can use the unitary that is given in the question and we can write

$$\hat{a}_{k_1}^\dagger \xrightarrow{U} \frac{1}{2}((1 + e^{i\varphi})\hat{a}_{k_1}^\dagger + (1 - e^{i\varphi})\hat{a}_{k_2}^\dagger), \quad (62)$$

$$\hat{a}_{k_2}^\dagger \xrightarrow{U} \frac{1}{2}((1 - e^{i\varphi})\hat{a}_{k_1}^\dagger + (1 + e^{i\varphi})\hat{a}_{k_2}^\dagger). \quad (63)$$

In the detectors, we are calculating the probability of having photons in that mode. So in detector D_1 we want to calculate the probability of having one photon in mode \mathbf{k}_1 . In other words, if our output state is $|\psi_{out}\rangle$ we can find the probability of having one photon in detector D_1 using the Born rule as follows.

$$p(n_1 = 1, n_2 = 0) = |\langle \psi_{out} | (|1\rangle_{\mathbf{k}_1} |0\rangle_{\mathbf{k}_2}) \rangle|^2 = |\langle \psi_{out} | 10 \rangle|^2 \quad (64)$$

So we just need to compute the output state.

$$|\psi_{in}\rangle = \hat{a}_{k_1}^\dagger |0\rangle_{\mathbf{k}_1} |0\rangle_{\mathbf{k}_2} = \hat{a}_{k_1}^\dagger |00\rangle \xrightarrow{U} \frac{1}{2}((1 + e^{i\varphi})\hat{a}_{k_1}^\dagger + (1 - e^{i\varphi})\hat{a}_{k_2}^\dagger) |00\rangle \quad (65)$$

$$= \frac{1}{2}(1 + e^{i\varphi})\hat{a}_{k_1}^\dagger |00\rangle + \frac{1}{2}(1 - e^{i\varphi})\hat{a}_{k_2}^\dagger |00\rangle \quad (66)$$

$$= \frac{1}{2}(1 + e^{i\varphi}) |10\rangle + \frac{1}{2}(1 - e^{i\varphi}) |01\rangle = |\psi_{out}\rangle \quad (67)$$

Now, for the first detector, we have

$$p(n_1 = 1, n_2 = 0) = |\langle \psi_{out} | 10 \rangle|^2 = \frac{1}{4}|1 + e^{i\varphi}|^2 = \frac{1}{2}(1 + \cos \varphi) = \cos^2 \frac{\varphi}{2} \quad (68)$$

and accordingly, the probability of having one photon in the detector D_2 is given by $p(n_1 = 0, n_2 = 1) = 1 - p(n_1 = 1, n_2 = 0) = \sin^2 \frac{\varphi}{2}$.

- (ii) Suppose one photon is input into mode \mathbf{k}_1 and one photon is input into mode \mathbf{k}_2 . What are the probabilities of finding 2, 1 or 0 photons at each of the detectors D_1 and D_2 ?

We suppose that we have one photon in the mode \mathbf{k}_1 and one photon in the mode \mathbf{k}_2 as input. So it means that our initial state is $\hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger |0\rangle_{\mathbf{k}_1} |0\rangle_{\mathbf{k}_2}$. To find the final state after the effect of beamsplitters and the phase shifter, we work in the Heisenberg picture and find the evolution of the creation operator. So we just need to compute the output state.

$$|\psi_{in}\rangle = \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger |0\rangle_{\mathbf{k}_1} |0\rangle_{\mathbf{k}_2} = \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger |00\rangle \quad (69)$$

$$\xrightarrow{U} \frac{1}{4}((1 + e^{i\varphi})\hat{a}_{k_1}^\dagger + (1 - e^{i\varphi})\hat{a}_{k_2}^\dagger)((1 - e^{i\varphi})\hat{a}_{k_1}^\dagger + (1 + e^{i\varphi})\hat{a}_{k_2}^\dagger) |00\rangle \quad (70)$$

$$= \frac{1}{4} \left((1 - e^{2i\varphi})\hat{a}_{k_1}^\dagger \hat{a}_{k_1}^\dagger + 2(1 + e^{2i\varphi})\hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger + (1 - e^{2i\varphi})\hat{a}_{k_2}^\dagger \hat{a}_{k_2}^\dagger \right) |00\rangle \quad (71)$$

$$= \frac{\sqrt{2}}{4}(1 - e^{2i\varphi}) |20\rangle + \frac{1}{2}(1 + e^{2i\varphi}) |11\rangle + \frac{\sqrt{2}}{4}(1 - e^{2i\varphi}) |02\rangle = |\psi_{out}\rangle \quad (72)$$

Where in the last line we use the fact that $a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$.

Now we want to compute the following probabilities at the detectors.

$$p(n_1 = 0, n_2 = 2) = |\langle \psi_{out} | (|0\rangle_{\mathbf{k}_1} |2\rangle_{\mathbf{k}_2}) \rangle|^2 = |\langle \psi_{out} | 02 \rangle|^2 \quad (73)$$

$$p(n_1 = 1, n_2 = 1) = |\langle \psi_{out} | (|1\rangle_{\mathbf{k}_1} |1\rangle_{\mathbf{k}_2}) \rangle|^2 = |\langle \psi_{out} | 11 \rangle|^2 \quad (74)$$

$$p(n_1 = 2, n_2 = 0) = |\langle \psi_{out} | (|2\rangle_{\mathbf{k}_1} |0\rangle_{\mathbf{k}_2}) \rangle|^2 = |\langle \psi_{out} | 20 \rangle|^2 \quad (75)$$

Now, using $|\psi_{out}\rangle$, we can compute the probabilities.

$$p(n_1 = 0, n_2 = 2) = |\langle\psi_{out}|02\rangle|^2 = \frac{1}{8}|1 - e^{2i\varphi}|^2 = \frac{1}{2}\sin^2(\varphi) \quad (76)$$

$$p(n_1 = 1, n_2 = 1) = |\langle\psi_{out}|11\rangle|^2 = \frac{1}{4}|1 + e^{2i\varphi}|^2 = \cos^2(\varphi) \quad (77)$$

$$p(n_1 = 2, n_2 = 0) = |\langle\psi_{out}|20\rangle|^2 = \frac{1}{8}|1 - e^{2i\varphi}|^2 = \frac{1}{2}\sin^2(\varphi) \quad (78)$$

- (iii) For each of the cases in (i) and (ii), imagine that just after the phase shifter a detector D_3 was inserted. Assuming it does not detect any photons, how are the probabilities of finding the photons at detectors D_1 and D_2 affected?

If we don't detect a photon at D_3 , this means the state before entering the second beamsplitter is given as $\hat{a}_{k_1}^\dagger |00\rangle$ for one-photon case and $\hat{a}_{k_1}^\dagger \hat{a}_{k_1}^\dagger |00\rangle$ for two-photon case. After passing through the second beamsplitter, we can write the output states and then compute the probabilities for these two cases.

— One Photon : In this case the output state is

$$|\psi_{in}\rangle = \hat{a}_{k_1}^\dagger |00\rangle \xrightarrow{BS2} \frac{1}{\sqrt{2}}(\hat{a}_{k_1}^\dagger + \hat{a}_{k_2}^\dagger) |00\rangle \quad (79)$$

$$= \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \quad (80)$$

Then the probabilities are $p(n_1 = 1, n_2 = 0) = p(n_1 = 0, n_2 = 1) = 1/2$.

— Two Photons : In this case the output state is

$$|\psi_{in}\rangle = \hat{a}_{k_1}^\dagger \hat{a}_{k_1}^\dagger |00\rangle \xrightarrow{BS2} \frac{1}{2}(\hat{a}_{k_1}^\dagger + \hat{a}_{k_2}^\dagger)(\hat{a}_{k_1}^\dagger + \hat{a}_{k_2}^\dagger) |00\rangle \quad (81)$$

$$= \frac{1}{2}(|20\rangle + 2|11\rangle + |02\rangle) \quad (82)$$

Then the probabilities are $p(n_1 = 2, n_2 = 0) = p(n_1 = 0, n_2 = 2) = 1/4$ and $p(n_1 = 1, n_2 = 1) = 1/2$.

- (iv) What is the relationship between this setup and the two-slit experiment?

In this part, the probability results are similar to the case in the two-slit experiment when we had a delayed quantum eraser. Here we do not see interference and we have a probabilistic mixture.